Characteristic Impedances of the Slotted Coaxial Line*

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Summary—**The characteristic impedance for the two possible TEM modes is calculated for a slotted coaxial line whose outer walls have a zero thickness. Conformal mapping is used in the calculations. The characteristic impedance for a slotted coaxial line is calculated in an approximate way for outer wall thickness different from zero.**

Introduction

As is known, in a cylindrical system of conductors having the cross section in Fig. 1(a) two modes of TEM waves can propagate. These modes correspond to the solution of the Laplace equation in the plane of the cross section for the boundary values shown in Fig. 1(b) and 1(c).

The wave impedance of the first of these modes was calculated by Collin[1] who used a variational method of analysis in which he neglected the thickness of the outer walls.

Bochenek[2] calculated the wave impedance for this first mode by using a conformal mapping method in which he neglected the inside conductor, but took into account the thickness of the outer walls.

The method of conformal mapping leads to the determination of the wave impedance in the previous case as well; *i.e.*, when we neglect the thickness of the outer walls and take into account the central conductor, the mapping can be performed in an exact way. At the same time, by making simplifying assumptions which do not give rise to any reservations in cases of practical interest, the approximate value of the impedance may be found without neglecting either the central conductor or the thickness of the walls.

In the case of the other mode of transmission [Fig. 1(c)] the conformal mapping method also leads to the determination of the impedance.

The subject of this report is to present the results of an analysis made along these lines and to compare them with the results published in both of the above-mentioned articles.

Mode of the First Type

The boundary problem for Laplace's equation [Fig. 1(b)] is transformed successively, by means of conformal mapping,[3] into boundary problems I, II, III

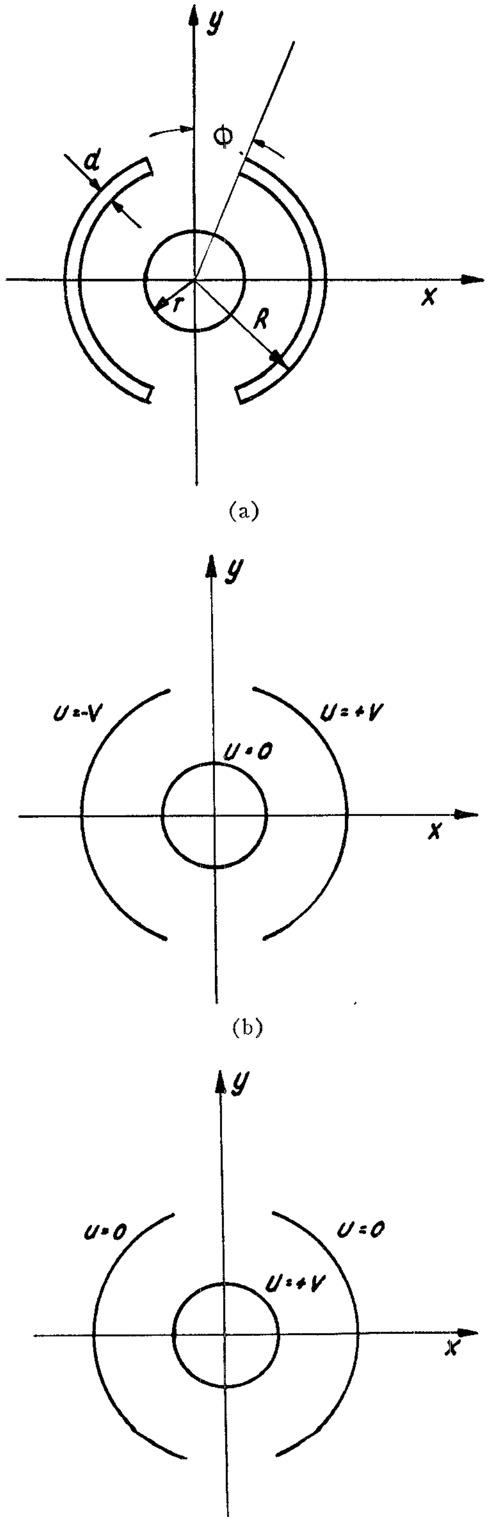

Fig. 1—System of conductors and the TEM modes examined. (a) Cross section of the system of conductors, (b) first transmission mode, (c) second transmission mode.

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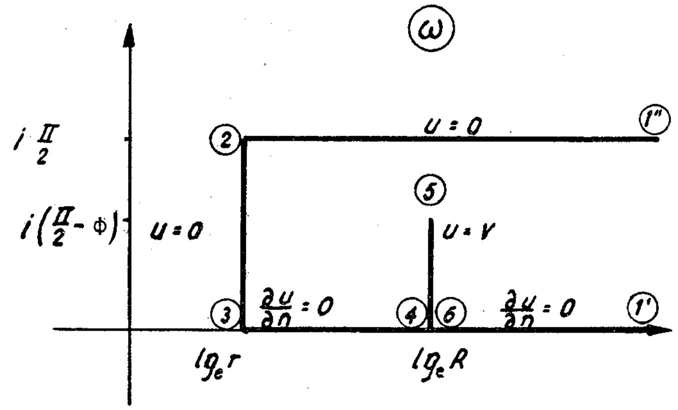
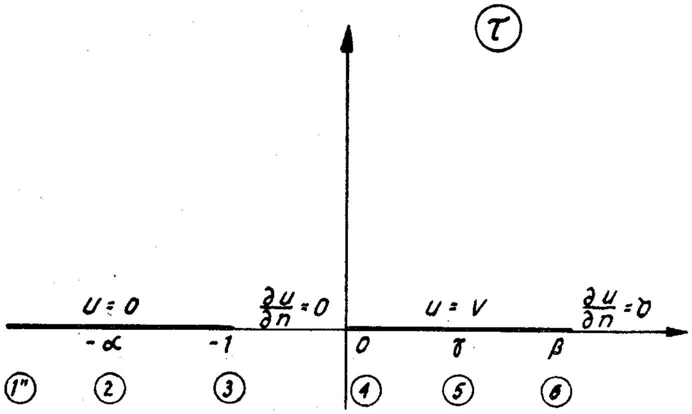
[1] R. E. Collin, "The characteristic impedance of a slotted coaxial line," IRE Trans., vol. MTT-4, pp. 4–8; January, 1956.

[2] K. Bochenek, "Impedancja falowa linii występującej w jednym z rodzajów symetryzatora," *Arch. Elektrotech.*, vol. 4, pp. 135–147; 1956.

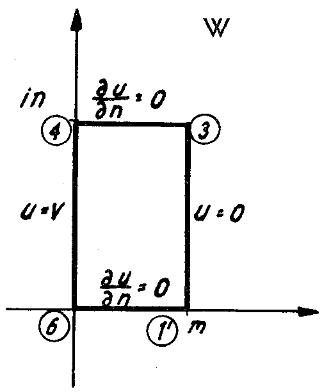
[3] E. Weber, "Electromagnetic Fields, Theory and Applications," John Wiley & Sons, Inc., New York, N. Y.; 1950.



(a)



(b)



(c)

Fig. 2—Boundary problems obtained by successive conformal mappings. Arabic numerals in the circles refer to points corresponding to each other. The boundary conditions for the function u are also given. (a) Boundary problem I—plane ω, (b) boundary problem II—plane τ, (c) boundary problem III—plane W.

[Fig. 2(a)–2(c)]. The splitting into successive transformations permits a better orientation as regards their character.

The first mapping has the form

$$\omega = \lg_e z. \tag{1}$$

The second is the Christoffel-Schwarz integral:

$$\omega = \frac{1}{2}\int_0^\tau \frac{(t-\gamma)dt}{\sqrt{(t-\beta)(t-0)(t+1)(t+\alpha)}} + \lg_e R. \tag{2}$$

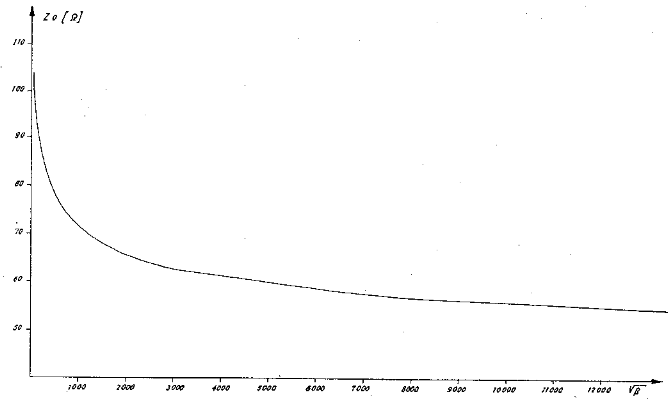


Fig. 3—Characteristic impedance as a function of the parameter β.

In the system corresponding to boundary problem II the wave impedance is defined by only one parameter β. The value of this impedance is readily found by means of the third mapping, which is also of the Christoffel-Schwarz type, and which reduces this system to the system of a plane ideal line (without boundary distortions). This mapping has the form

$$\Psi = C_1\int_0^\tau \frac{dt}{\sqrt{(t-0)(t-\beta)(t+1)}} + in. \tag{3}$$

A simple calculation gives a formula for the wave impedance as a function of β:

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon^0}}\,\frac{K(k)}{K(k')} \tag{4}$$

where

$$k = \sqrt{\frac{1}{1+\beta}} \qquad k' = \sqrt{\frac{\beta}{1+\beta}}$$

and which is given in Fig. 3. $K(k)$ and $K(k')$ are complete elliptic integrals of the first kind.

The finding of β as a function of the angle ϕ and the ratio of the radii R/r of the line presents a basic difficulty. To find this relation, mapping II should be studied more closely.

To the arbitrarily selected values of the parameters α, β, and γ there corresponds the region ω in which points 4 and 6 may not coincide, as in the case in Fig. 4. Hence, in the case of interest to us, α, β, and γ should be determined by means of the three conditions: 1) $\lg_e R/r$ has the selected value, 2) the angle ϕ has the selected value, 3) points 4 and 6 coincide.

These conditions written in analytical form appear as follows:

$$\lg_e\frac{R}{r} = \frac{1}{2}\int_{-1}^{0}\frac{(t-\gamma)dt}{\sqrt{(t-\beta)(t-0)(t+1)(t+\alpha)}} \tag{5}$$

$$i\left(\frac{\pi}{2}-\phi\right) = \frac{1}{2}\int_0^\gamma\frac{(t-\gamma)dt}{\sqrt{(t-\beta)(t-0)(t+1)(t+\alpha)}} \tag{6}$$

$$i\frac{\pi}{2} = \frac{1}{2}\int_{-1}^{-\alpha}\frac{(t-\gamma)dt}{\sqrt{(t-\beta)(t-0)(t+1)(t+\alpha)}}. \tag{7}$$

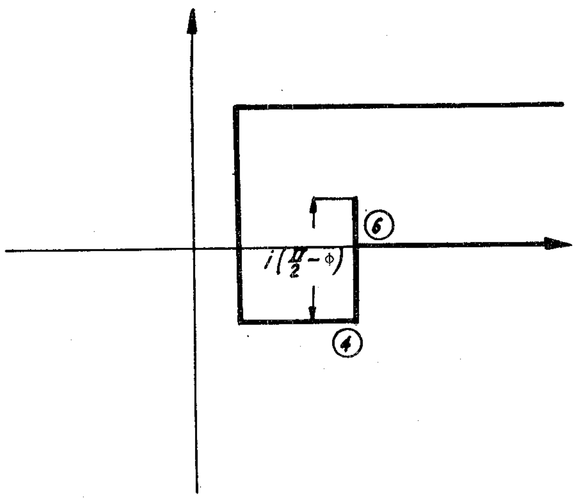


Fig. 4—Displacement of points 4 and 6.

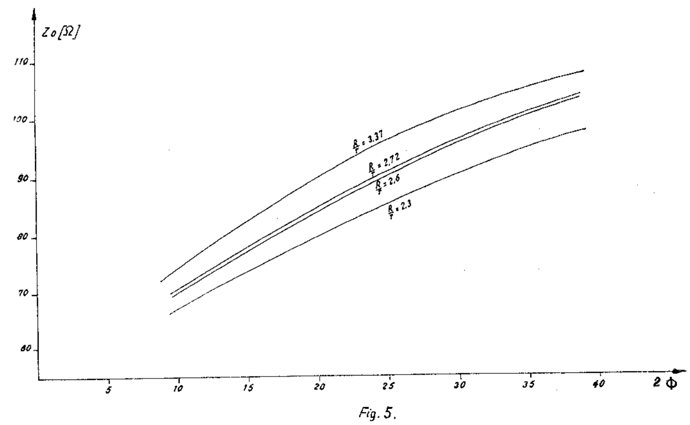


Fig. 5—Characteristic impedance as a function of the angle 2ϕ for various values of the ratios R/r. $d=0$.

With the help of classical transformations we express these relations by means of elliptic integrals:[4]

$$\lg_e \frac{r}{R} = \frac{1}{\sqrt{\alpha(\beta+1)}}\left[(\alpha-1)\Pi(\rho_1, k) - (\alpha+\gamma)K(k')\right] \quad (8)$$

$$k' = \sqrt{\frac{\beta+\alpha}{(\beta+1)\alpha}} \qquad \rho_1 = -\frac{1}{\alpha}$$

$$\frac{\pi}{2} - \phi = \frac{1}{\sqrt{\alpha(\beta+1)}}\left[(1+\gamma)F(\Psi, k) - \Pi(\Psi, \rho_2, k)\right] \quad (9)$$

$$k = \sqrt{\frac{\beta(\alpha-1)}{(\beta+1)\alpha}} \qquad \rho_2 = -\frac{\beta}{\beta+1}$$

$$\Psi = \arcsin\sqrt{\frac{(\beta+1)\gamma}{\beta(\gamma+1)}}$$

$$\frac{\pi}{2} = \frac{1}{\sqrt{\alpha(\beta+1)}}\left[(\gamma-\beta)K(k) + (\beta+\alpha)\Pi(\rho_3, k)\right] \quad (10)$$

$$k = \sqrt{\frac{\beta(\alpha-1)}{(\beta+1)\alpha}} \qquad \rho_3 = -\frac{\alpha-1}{\beta+1}.$$

In these equations $K(k)$ is the complete elliptic integral of the first kind, $F(\Psi, k)$ is the incomplete elliptic integral of the first kind, $\Pi(\rho, k)$ is the complete elliptic integral of the third kind, and $\Pi(\Psi, \rho, k)$ is the incomplete elliptic integral of the third kind. The mathematical procedure leading to the finding of the values of β, α, and γ (unfortunately, the latter two parameters must also be evaluated) corresponding to the values of R/r and ϕ which interest us is presented in Appendix I.

The values obtained for the impedance as a function of the angle 2ϕ are shown in Fig. 5 for ratios $R/r=3.37$, 2.72, 2.6, 2.3.

[4] See; *e.g.*, P. Byrd and M. Friedman, "Handbook of Elliptic Integrals for Engineers and Physicists," Springer-Verlag, Berlin, Germany; 1954.
W. Gröbner and N. Hofreiter, "Integraltafel," Springer-Verlag, Berlin, Germany; 1949.

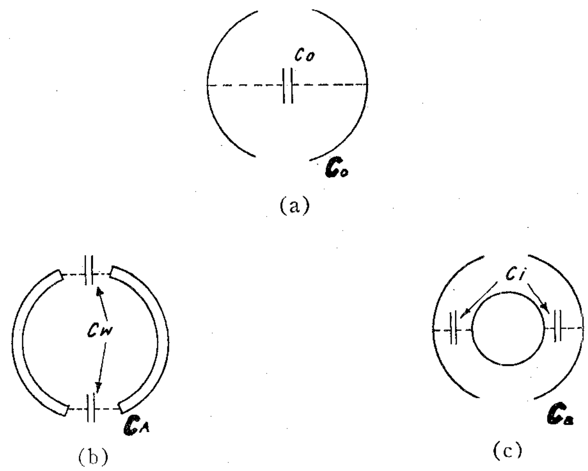


Fig. 6—Capacity distribution. (a) C_0—capacity between the thin outer conductors, (b) C_A—total capacity of the system without the inner conductor. $C_W = C_A - C_0$ capacity introduced by thickening the walls. (c) C_B—total capacity of the system with thin outer walls. $C_i = C_B - C_0$ = capacity introduced by the central conductor.

When the thickness of the walls is not neglected; *i.e.*, the line shown in Fig. 1(a), the capacity introduced by the central conductor and the capacity introduced by the thickening of the walls may be considered as adding to each other (Fig. 6) as long as the increase in wall thickness does not affect the field distribution in the vicinity of the central conductor and the introduction of the central conductor does not affect the field distribution in the gaps between the outer conductors. The greater the radius of the outer conductor in relation to both the radius of the central conductor and the thickness of the walls of the outer conductor, the more correct this assumption seems.

The impedance calculations presented below are based on the determination of the additional capacity introduced into the system by the thickening of the walls of the outer conductor on the basis of formulas derived by Bochenek[2] and the additional capacity introduced by the central conductor on the basis of the results obtained here.

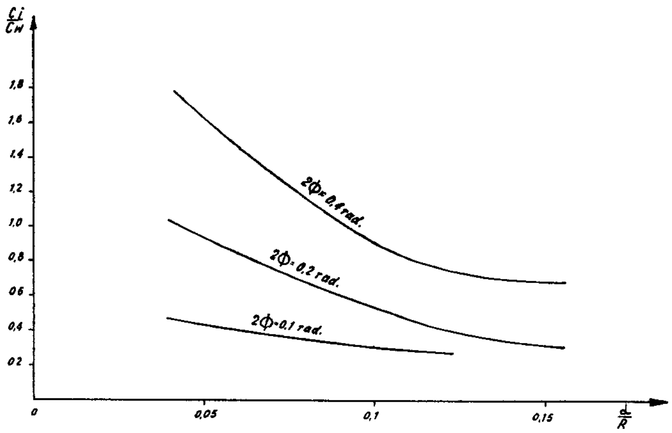


Fig. 7—Ratio of capacity introduced by the inner conductor to the capacity introduced by thickening the walls for various angles (2ϕ).

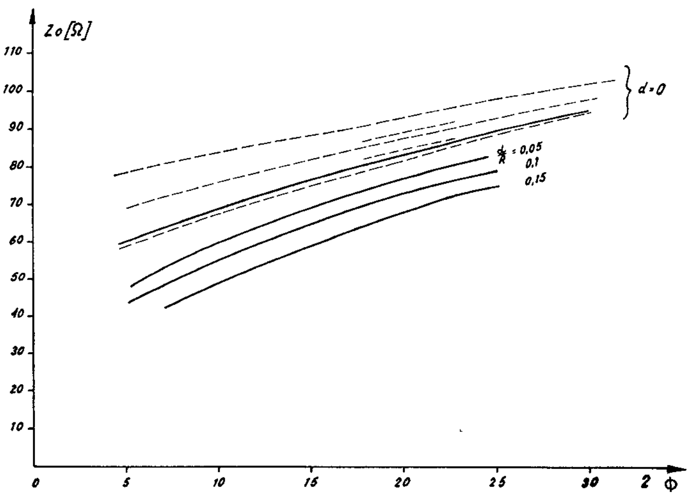


Fig. 8—Solid line—characteristic impedance as a function of the angle 2ϕ for $R/r=2.6$ and the various values of d/R. Dotted lines—approximations obtained by a variational method by Collin[1] for $d=0$.

The ratio of the capacity introduced by the inner conductor C_i to the capacity introduced by the thickening of the walls C_w is represented in Fig. 7. This curve gives an idea of the influence of both factors on the values of the impedance.

The next curve (Fig. 8) represents the impedance for the ratio $R/r=2.6$, wall thickness $d/R=0$, 0.05, 0.1, and 0.15 as a function of the angle 2ϕ. For comparison, the curves obtained by Collin[1] for the case $d=0$ are also given in the figure.

Fig. 9 presents a comparison of the approximate value of the characteristic impedance obtained by Collin[1] by a variational method for $r=0$ and $d=0$ with the exact value obtained by means of homographic mapping.

Mode of the Second Type

This mode [Fig. 1(c)] corresponds to the basic mode in a coaxial line distorted by the slotting of the outer conductor. It seems of interest to investigate the effect of this distortion on the values of the wave impedance.

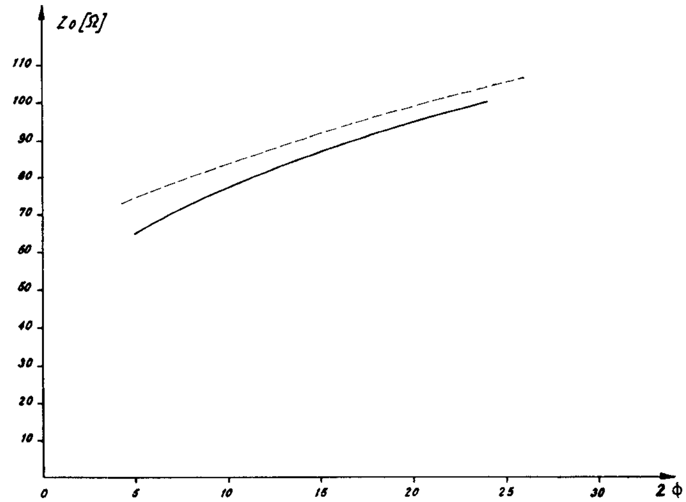


Fig. 9—Solid line—characteristic impedance as a function of angle 2ϕ for $R/r=2.6$ and $d=0$. Dotted line—approximation obtained by a variational method. (See reference 1.)

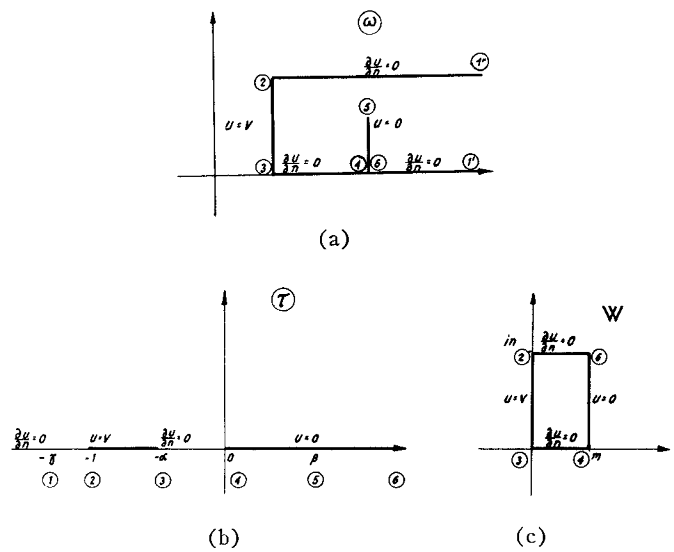


Fig. 10—Boundary problems obtained by successive conformal mappings. Arabic numerals in the circles denote points corresponding to each other. Boundary conditions for the function u are also given. (a) Boundary conditions problem I—plane ω, (b) boundary problem II—plane τ, (c) boundary problem III—plane W.

As in the preceding case, we use three successive conformal mappings:

$$\omega = \lg_e z \tag{11}$$

$$\omega = -i\,\frac{\sqrt{(\gamma-1)(\gamma-\alpha)\gamma}}{2(\gamma+\beta)} \cdot \int_0^{\tau} \frac{(t-\beta)dt}{(t+\gamma)\sqrt{(t+0)(t+\alpha)(t+1)}} + \lg_e R; \tag{12}$$

$$\omega = C_1 \int_0^{\tau} \frac{dt}{\sqrt{(t-0)(t+\alpha)(t+1)}} + m. \tag{13}$$

The boundary problems corresponding to these mappings are shown in Fig. 10.

The impedance in system II [Fig. 10(b)] is defined by only one parameter α. Its value is found by means of

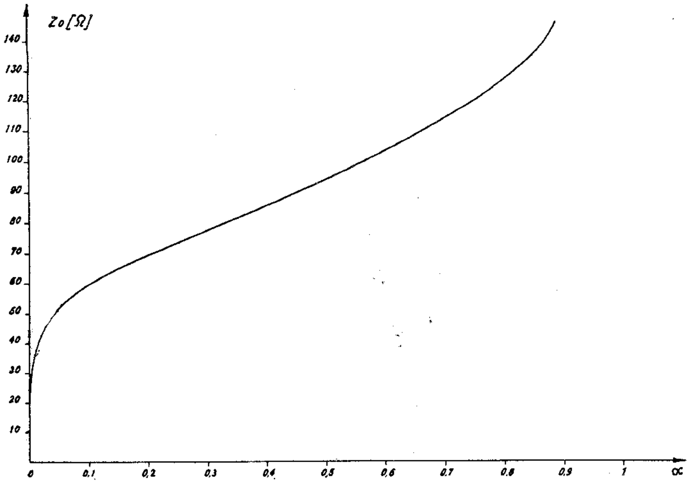


Fig. 11—Characteristic impedance as a function of parameter α.

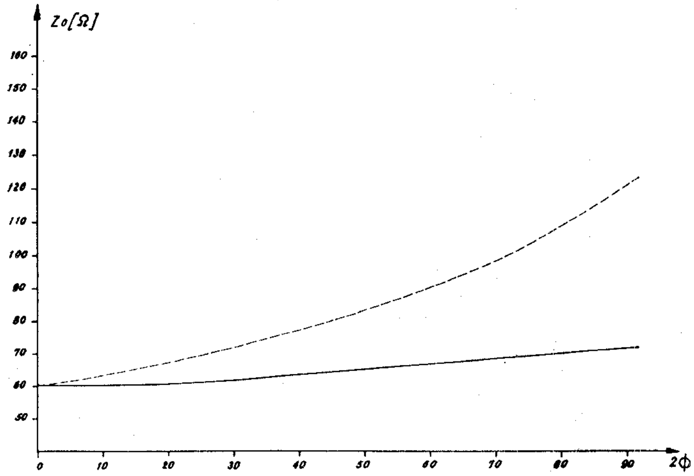


Fig. 12—Solid line—characteristic impedance of the second mode as a function of the angle 2ϕ. Dotted line—result of the simplified calculation (see text).

mapping in the system III, identically as in the preceding case.

The dependence of the impedance on α is expressed by the formula:

$$Z_0 = \frac{1}{4}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{K(k)}{K(k')} \tag{14}$$

where

$$k = \sqrt{\alpha} \qquad k' = \sqrt{1-\alpha}.$$

This relation is shown in Fig. 11.

Similarly as in the previous case, the calculations lead us to the following system of equations giving the relation between the parameters α, β, γ and the quantities R/r and ϕ:

$$\lg_e \frac{R}{r} = \frac{\sqrt{\gamma(\gamma-\alpha)}}{(\gamma+\beta)\sqrt{\gamma-1}}\left[(1+\beta)K(k) - \frac{1-\alpha}{\gamma-\alpha}(\gamma+\beta)\Pi(\rho_1, k)\right]; \tag{15}$$

$$k = \sqrt{\alpha} \qquad \rho_1 = -\frac{\alpha(\gamma-1)}{\gamma-\alpha}$$

$$\frac{\pi}{2} - \phi = \frac{\sqrt{(\gamma-1)\gamma}}{(\gamma+\beta)\sqrt{\gamma-\alpha}}\left[(\beta+\alpha)F(\Psi, k') - \frac{(\beta+\gamma)\alpha}{\gamma}\Pi(\Psi, \rho_2, k')\right]; \tag{16}$$

$$k' = \sqrt{1-\alpha} \qquad \rho_2 = -\frac{\gamma-\alpha}{\gamma}$$

$$\Psi = \arcsin\sqrt{\frac{\beta}{\beta+\alpha}}$$

$$\frac{\pi}{2} = \frac{\sqrt{(\gamma-1)(\gamma-\alpha)\gamma}}{\gamma+\beta}\left[\frac{\beta+\gamma}{\gamma-1}\Pi(\rho_3, k') - K(k')\right] \tag{17}$$

$$k' = \sqrt{1-\alpha} \qquad \rho_3 = -\frac{\alpha-1}{\gamma-1}\,.$$

The procedure for determining α, β, and γ for the values of the ratio R/r and the angle ϕ, of interest to us, is given in Appendix II.

The values obtained for the ratio $R/r=2.72$ as a function of the angle 2ϕ are given in Fig. 12. Also given in this figure is the curve of the impedance calculated by assuming that the capacity of the coaxial line changes proportionally to the angle subtended by the cross section of the outer conductor. As may be seen, this type of simplification of the impedance calculation gives results which depart considerably from reality; the actual impedance increases more slowly than the impedance calculated in the above approximative manner. This should have been expected when one considers that the field lines become more dense at the boundary of the gap.

In the case of the type of wave being considered here, it seems that the influence of the thickness of the outer conductor on the impedance will be considerably smaller than in the preceding case since both parts of the outer conductor are at the same potential.

Appendix I

Calculation of the First Transmission Mode

In the range of the values of the parameters of interest to us, (8)–(10) can be replaced, for purposes of calculation, by simplified formulas which are derived from the assumption that

$$\beta \gg 1, \quad \beta \gg \gamma, \quad \text{and} \quad \beta \gg \alpha.$$

$$\lg_e \frac{r}{R} \cong \frac{1}{\sqrt{\alpha\cdot\beta}}\left[\alpha E(k') - (\alpha+\gamma)K(k')\right] \tag{18}$$

$$\frac{\pi}{2} - \phi \cong \frac{1}{\sqrt{\alpha\cdot\beta}}\left[\gamma F(\Psi, k) + \alpha E(\Psi, k) - \sqrt{\alpha}\,\mathrm{tg}\Psi\right] \tag{19}$$

$$\frac{\pi}{2} \cong \frac{1}{\sqrt{\alpha\cdot\beta}}\left[(\gamma+\alpha)K(k)\right]. \tag{20}$$

$E(k')$ is the complete elliptic integral of the second kind.

$$k = \sqrt{\frac{\alpha - 1}{\alpha}} \qquad k' = \sqrt{\frac{1}{\alpha}}$$

$$\rho = -1 \qquad \Psi = \arcsin \sqrt{\frac{\gamma}{\gamma + 1}}\,.$$

For the range of the angle ϕ and ratios R/r, of interest to us, these assumptions are correct and can be directly verified by substituting the results obtained into the exact formulas.[5]

We insert in (18) the expression for $\gamma+\alpha$ obtained from (20) and we obtain (21)

$$\sqrt{\beta} = \frac{\sqrt{\alpha}E(k')}{\lg_e \frac{r}{R} + \frac{\pi}{2}\frac{K(k')}{K(k)}}\,. \tag{21}$$

The system (19) through (21) obtained in this way is a convenient point of departure for the numerical calculations.

The computational procedure is as follows: we assume a value for R/r and a value for α. In this way the parameters of the system are uniquely defined. Only the corresponding value of the angle ϕ has to be found. To do this, we insert the values taken for α and R/r into (21) and determine β from this equation. We insert the value found for β and the value taken for α into (20) and obtain γ. We substitute the values of α, β, and γ into (19) and obtain the angle ϕ. By taking different values of α we can change ϕ in the interval of interest to us. The value of β obtained in the course of the computations uniquely determines the impedance (4),

$$\text{for } \beta \gg 1: \qquad Z_0 = \frac{591.4}{\lg_e 4\sqrt{\beta}}\,.$$

As may be seen, the calculations are quite simple, since they reduce to looking up the elliptic integrals in the tables and to making some arithmetical calculations.

[5] The correctness of the assumption that $\gamma \gg 1$ can also be checked in the following way: β is a parameter which uniquely determines the impedance; with an increase in β the impedance decreases. The impedance for the system with the central conductor is smaller (and therefore β is greater) than the impedance (the value of β) for the system without the central conductor. For the system without the central conductor, the corresponding value of impedance can be readily found by means of homographic mapping.

Appendix II

Calculation of the Second Transmission Mode

In the ranges of the angle ϕ and the ratios R/r of interest to us, $\gamma \gg 1$, and therefore (15) through (17) can be replaced for purposes of calculation by simplified formulas:

$$\lg_e \frac{R}{r} = \frac{\sqrt{\gamma}}{\gamma + \beta} \cdot \left[\frac{\gamma[K(k) - E(k)] + \beta[\gamma K(k) - \alpha K(k) - E(k)]}{\gamma}\right]; \tag{22}$$

$$\frac{\pi}{2} - \phi = \frac{\sqrt{\gamma}}{\gamma + \beta}\left[(\beta + \alpha)F(\Psi, k') - \frac{\gamma + \beta}{\gamma}\cdot \alpha \Pi(\Psi, \rho, k')\right]; \tag{23}$$

$$\frac{\pi}{2} = \frac{\sqrt{\gamma}}{\gamma + \beta} K(k')[1 + \beta] \tag{24}$$

$$k = \sqrt{\alpha} \qquad k' = \sqrt{1 - \alpha} \qquad \rho = -1$$

$$\Psi = \arcsin \sqrt{\frac{\beta}{\beta + \alpha}}\,.$$

We insert in (22) the expression for β obtained from (24) and we obtain (25):

$$\gamma\left[K(k')\lg_e \frac{R}{r} - \frac{\pi}{2}K(k)\right] + \sqrt{\gamma}E(k)K(k') + \frac{\pi}{2}K(k)(1 + \alpha) - K(k')\lg_e \frac{R}{r} = 0. \tag{25}$$

The calculations are made as follows: we assume a value for R/r and a value for α. To find the angle ϕ, we insert the values taken for R/r and α into (25) and from this equation determine γ. We insert the value found for γ and that taken for α into (24) and obtain β. We put the values of α, β, and γ into (23) and obtain the angle ϕ. By taking various values for α we can change ϕ into the interval in which we are interested. The value of α uniquely determines the impedance (14).

Acknowledgment

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